JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **33**, No. 2, May 2020
http://dx.doi.org/10.14403/jcms.2020.33.2.205

A NEW GENERALIZED QUARTIC FUNCTIONAL EQUATION AND ITS STABILITY PROBLEMS

Heejeong Koh*

Abstract. We will introduce a new type of quartic functional equation and then investigate the stability for a quartic functional equation in a convex modular space.

1. Introduction

The stability problem for a functional equation was first posed by Ulam [23] in the context of the stability of group homomorphisms. In the next year, Hyers [6] gave a partial answer to the question of Ulam. Subsequently, Hyers' theorem was generalized in various directions. The first author who generalized Hyers' theorem to the case of unbounded control functions was Aoki [1]. Rassias [20] succeeded in extending the result of Hyers' by weakening the condition for the Cauchy difference. Rassias' paper [20] has been very influential provided in the development of Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. In 1996, Isac and Rassias [7] were first to provide applications of new fixed point theorems for the proof of the stability of functional equations. By using fixed point methods the stability problems of several functional equations over various normed spaces have been extensively investigated by a number of authors; see [4], [3], [17] and [18]. In particular, Rassias [19] introduced the quartic functional equation

$$f(x+2y)+f(x-2y)+6f(x)=4f(x+y)+4f(x-y)+24f(y)\,. \tag{1.1}$$

It is easy to see that $f(x)=x^4$ is a solution of (1.1) by virtue of the identity

$$(x+2y)^4+(x-2y)^4+6x^4=4(x+y)^4+4(x-y)^4+24y^4\,. \tag{1.2}$$

Received September 20, 2019; Accepted February 28, 2020.
2010 Mathematics Subject Classification: 39B52.
Key words and phrases: quartic functional equation, modular space, convex modular, $\triangle_a$-condition, Hyers-Ulam-Rassias stability, fixed point theorem.

For this reason, (1.1) is called a quartic functional equation. Chung and Sahoo [5] determined the general solution of (1.1) without assuming any regularity conditions on the unknown function. In fact, they proved that the function $f:\mathbb{R}\to\mathbb{R}$ is a solution of (1.1) if and only if $f(x)=A(x,x,x,x)$, where the function $A:\mathbb{R}^4\to\mathbb{R}$ is symmetric and additive in each variable. Lee and Chung [9] introduced a quartic functional equation as follows:

$$\begin{aligned}f(ax+y)+f(ax-y)\\ =a^2f(x+y)+a^2f(x-y)+2a^2(a^2-1)f(x)-2(a^2-1)f(y)\,,\end{aligned}\tag{1.3}$$

for fixed integer a with $a\neq 0,\ \pm1$.

As we notice there are various definitions for the stability of the quartic functional equations, in this paper, we will introduce a new type of a generalized quartic functional equation as follows :

$$\begin{aligned}f(ax-by)+f(bx-ay)+\frac{ab}{2}(a-b)^2f(x+y)\\ =\frac{ab}{2}(a+b)^2f(x-y)+(a^2-b^2)^2[f(x)+f(y)]\end{aligned}\tag{1.4}$$

where a and b are integers with $a\neq b$ and $a,\,b\neq 0,\pm1$.

Definition 1.1. Let X be a linear space over a field $\mathbb{K}\,(\mathbb{R}$ or $\mathbb{C})$. A generalized functional $\rho : X\to[0,\,\infty]$ is called a modular if for any $x\,,y\in X$,

(M1) $\rho(x)=0$ if and only if $x=0$.
(M2) $\rho(\alpha x)=\rho(x)$ for all scalar α with $|\alpha|=1$.
(M3) $\rho(\alpha x+\beta y)\leq\rho(x)+\rho(y)$ for all scalars $\alpha\,,\beta\geq0$ with $\alpha+\beta=1$.

If (M3) is replaced by

(M4) $\rho(\alpha x+\beta y)\leq\alpha\rho(x)+\beta\rho(y)$ for all scalars $\alpha\,,\beta\geq0$ with $\alpha+\beta=1$, then the functional ρ is said to be a convex modular.

A modular ρ defines the following vector space:

$$X_\rho:=\{x\in X\mid\rho(\lambda x)\to0\text{ as }\lambda\to0\}$$

and we call X_ρ a modular space. Let ρ be a convex modular. The norm on the modular space X_ρ is defined by

$$||x||_\rho=\inf\Big\{\lambda>0\,|\,\rho\Big(\frac{x}{\lambda}\Big)\leq1\Big\}\,.$$

It is called the Luxemburg norm.

A modular ρ is said to satisfy the Δ_2-condition if there exists $k>0$ such that $\rho(2x)\leq k\rho(x)$ for all $x\in X_\rho$. We call the constant k a Δ_2-constant related to Δ_2-condition. Now, let $\{x_n\}$ be a sequence in X_ρ.

The sequence $\{x_n\}$ is ρ-convergent to a point $x \in X_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$. The sequence $\{x_n\}$ is called ρ-Cauchy if for any $\varepsilon > 0$ one has $\rho(x_n - x_m) < \varepsilon$ for sufficiently large $n, m \in \mathbb{N}$. Also, X_ρ is called ρ-complete if any ρ-Cauchy sequence is ρ-convergent to a point in X_ρ. The modular ρ has the Fatou property if and only if $\rho(x) \leq \liminf_{n\to\infty} \rho(x_n)$ whenever the sequence $\{x_n\}$ is ρ-convergent to x. The modular theory on linear spaces and the related modular theory on linear spaces have been established by Nakano [16]. Kim and Shin [11] investigated the stability problems of additive and quadratic functional equations in modular spaces.

In this paper, we will obtain a general solution of the generalized quartic functional equation (1.4) and then investigate the stability problems by using both the direct method and the fixed point method for the given generalized quartic functional equation in the modular space.

2. A solution for a generalized quartic functional equation

In this section let X and Y be real vector spaces. We will investigate the general solution of the functional equation (1.4)concerning n-additive symmetric mappings. The key concepts are found in [22] and [24].

THEOREM 2.1. *A mapping $f : X \to Y$ is a solution of the functional equation (1.4) if and only if f is of the form $f(x) = A^4(x)$ for all $x \in X$, where $A^4(x)$ is the diagonal of a 4-additive symmetric mapping $A_4 : X^4 \to Y$.*

Proof. Suppose f satisfies the functional equation (1.4). On letting $x = y = 0$ in the equation(1.4),

$$(2a^4 - 2a^2b^2 + 2b^4 - 2)f(0) = 0\,.$$

Hence $f(0) = 0$. On putting $y = 0$ in the equation (1.4), we have

$$f(ax) + f(bx) = a^4 f(x) + b^4 f(x) \tag{2.1}$$

for all $x \in X$. Also, on letting $x = 0$ in the equation (1.4) and then replacing y by $-x$, we get

$$f(bx) + f(ax) + \frac{ab}{2}(a-b)^2 f(-x) - \frac{ab}{2}(a+b)^2 f(x) - (a^2-b^2)^2 f(-x) = 0$$

for all $x \in X$. By using the equation (2.1), we have

$$(a^4 + b^4 - \frac{1}{2}a^3b - a^2b^2 - \frac{1}{2}ab^3)(f(-x) - f(x)) = 0\,.$$

That is, $f(x) = f(-x)$, for all $x \in X$. By Theorems 3.5 and 3.6 in [24], f is a generalized polynomial function of degree at most 4, that is, f is of the form

$$f(x) = A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x) \tag{2.2}$$

for all $x \in X$, where $A^0(x) = A^0$ is an arbitrary element of Y and $A^i(x)$ is the diagonal of an i-additive symmetric mapping $A_i : X^i \to Y$ for $i = 1, 2, 3, 4$. By $f(0) = 0$ and $f(-x) = f(x)$ for all $x \in X$, we get $A^0(x) = A^0 = 0$ and $A^1(x) = A^3(x) = 0$. Hence we have

$$f(x) = A^4(x) + A^2(x)\,,$$

for all $x \in X$. The equation (2.1) and $A^n(rx) = r^n A^n(x)$ for all $x \in X$ and all $r \in \mathbb{Q}$ imply that

$$a^4(A^4(x)+A^2(x))+b^4(A^4(x)+A^2(x)) = (a^4+b^4)A^4(x)+(a^2+b^2)A^2(x)$$

for all $x \in X$. Hence we may conclude that $A^2(x) = 0$. Thus $f(x) = A^4(x)$ for all $x \in X$, as desired.

Conversely, assume that $f(x) = A^4(x)$ for all $x \in X$, where $A^4(x)$ is the diagonal of a 4-additive symmetric mapping $A_4 : X^4 \to Y$. Note that

$$\begin{aligned} &A^4(qx + ry) \\ &= q^4A^4(x) + 4q^3rA^{3,1}(x, y) + 6q^2r^2A^{2,2}(x, y) + 4qr^3A^{1,3}(x, y) + r^4A^4(y) \\ &\quad c^sA^{s,t}(x, y) = A^{s,t}(cx, y)\,, \quad c^tA^{s,t}(x, y) = A^{s,t}(x, cy) \end{aligned}$$

where $1 \le s, t \le 3$ and $c \in \mathbb{Q}$. Thus we may conclude that f satisfies the equation (1.4). $\square$

Now, we call the mapping f a *generalized quartic mapping* if f satisfies the equation (1.4).

3. The Direct Method Approach

Throughout this section let V be a linear space and X_ρ a ρ-complete convex modular space unless otherwise stated. Now, we will state some basic properties as a remark to be used in this section.

Remark 3.1. 1. $\rho(\alpha x) \le \alpha\rho(x)$ for all $0 \le \alpha \le 1$.
2. $\rho(\sum_{j=1}^n \alpha_j x_j) \le \sum_{j=1}^n \alpha_j \rho(x_j)$ for all $\alpha_j \ge 0$, where $\sum_{j=1}^n \alpha_j \le 1$.

LEMMA 3.2. *Let X be a linear space over a field $\mathbb{K}\,(\mathbb{R}$ or $\mathbb{C})$. Suppose X satisfies the $\triangle_2$-condition with $\triangle_2$-constant k. Then for each $a \in \mathbb{K}$ with $|a| > 1$ there exists a constant k_a such that $\rho(ax) \le k_a\rho(x)$ for all $x \in X$.*

Proof. Since $|a| > 1$, then there exists a positive integer n such that $2^{n-1} < |a| \le 2^n$. Hence we have

$$\begin{aligned}\rho(ax) = \rho(\frac{a}{|a|}|a|x) = \rho(|a|x) &= \rho(\frac{|a|}{2^n}2^n x) \\ &\le \frac{|a|}{2^n}\rho(2^n x) \le \frac{|a|}{2^n}k^n\rho(x) = k_a\rho(x)\end{aligned}$$

where $k_a = \frac{|a|}{2^n}k^n$. □

DEFINITION 3.3. Let $a \ge 2$ be an integer. A modular ρ is said to satisfy the Δ_a-condition if there exists $k_a > 0$ such that $\rho(ax) \le k_a\rho(x)$ for all $x \in X_\rho$. We call the constant k_a a Δ_a-constant related to the Δ_a-condition and a.

LEMMA 3.4. *Let $a \ge 2$ be a fixed integer. Let X be a linear space over a field $\mathbb{K}\,(\mathbb{R}$ or $\mathbb{C})$. Suppose X satisfies the $\triangle_a$-condition with $\triangle_a$-constant k_a. Then the $\triangle_a$-constant k_a is greater than equal to a.*

Proof. If $k_a < a$, then $\rho(x) \le \frac{1}{a}\rho(ax) \le \frac{k_a}{a}\rho(x) < \rho(x)$, for all $x \in X$. It is a contradiction. □

For a given function $f : V \to X_\rho$ and a fixed integer $a \ge 2$ let

$$\begin{aligned}D_af(x, y) := f(ax - y) + f(x - ay) + \frac{a}{2}(a-1)^2 f(x+y) \\ -\frac{a}{2}(a+1)^2 f(x-y) - (a^2-1)^2[f(x) + f(y)]\end{aligned}$$

THEOREM 3.5. *Let $a \ge 2$ be an integer. Suppose X_ρ satisfies the $\triangle_a$-condition with $\triangle_a$-constant k_a. If there exists a function $\phi : V \to [0, \infty)$ for which a mapping $f : V \to X_\rho$ satisfies*

$$\rho(D_af(x, y)) \le \phi(x, y) \tag{3.1}$$

$$\lim_{n\to\infty} k_a^{4n}\phi\Big(\frac{x}{a^n}, \frac{y}{a^n}\Big) = 0 \text{ and } \sum_{j=1}^{\infty}\Big(\frac{k_a^5}{a}\Big)^j\phi\Big(\frac{x}{a^j}, \frac{y}{a^j}\Big) < \infty \tag{3.2}$$

for all x, $y \in V$, then there exists a unique generalized quartic mapping $Q : V \to X_\rho$ defined by $Q(x) = \rho - \lim_{n\to\infty} a^{4n} f\Big(\frac{x}{a^n}\Big)$ and

$$\rho\Big(f(x) - Q(x)\Big) \le \frac{1}{a\,k_a^3} \sum_{j=1}^{\infty} \Big(\frac{k_a^5}{a}\Big)^j \phi\Big(\frac{x}{a^j},\, 0\Big) \tag{3.3}$$

for all $x \in V$.

Proof. On taking $x = y = 0$ in the inequality (3.1), we have $\rho(2a^2(1-a^2)f(0)) \le \phi(0, 0)$. The equation (3.2) implies that $f(0) = 0$. Now, replacing x and y by $\frac{x}{a}$ and 0 respectively, we have

$$\rho\Big(f(x) - a^4 f\Big(\frac{x}{a}\Big)\Big) \le \phi\Big(\frac{x}{a}, 0\Big) \tag{3.4}$$

for all $x \in V$. For any positive integer n, the $\triangle_a$-condition and the Remark 3.1 imply that

$$\begin{aligned}\rho\Big(f(x) - a^{4n} f\Big(\frac{x}{a^n}\Big)\Big) &= \rho\Big(\sum_{j=1}^{n} \frac{1}{a^j}\Big(a^{5j-4} f\Big(\frac{x}{a^{j-1}}\Big) - a^{5j} f\Big(\frac{x}{a^j}\Big)\Big)\Big)\\ &\le \frac{1}{k_a^4} \sum_{j=1}^{n} \Big(\frac{k_a^5}{a}\Big)^j \phi\Big(\frac{x}{a^j},\, 0\Big)\end{aligned}$$

for all $x \in V$. For all positive integers n and m with $n \ge m$, we get

$$\begin{aligned}\rho\Big(a^{4n} f\Big(\frac{x}{a^n}\Big) - a^{4m} f\Big(\frac{x}{a^m}\Big)\Big) &\le k_a^{4m} \rho\Big(a^{4(n-m)} f\Big(\frac{x}{a^n}\Big) - f\Big(\frac{x}{a^m}\Big)\Big)\\ &\le \frac{1}{k_a^4} k_a^{4m} \sum_{j=1}^{n-m} \Big(\frac{k_a^5}{a}\Big)^j \phi\Big(\frac{x}{a^{m+j}},\, 0\Big)\\ &\le \frac{1}{k_a^4} \Big(\frac{a}{k_a}\Big)^m \sum_{j=m+1}^{n} \Big(\frac{k_a^5}{a}\Big)^j \phi\Big(\frac{x}{a^j},\, 0\Big)\end{aligned}$$

for all $x \in V$. The last part of the above inequalities tends to zero as $m \to \infty$. Hence the sequence $\{a^{4n} f\Big(\frac{x}{a^n}\Big)\}$ is a ρ-Cauchy sequence in the ρ-complete convex modular space. This means that the sequence $\{a^{4n} f\Big(\frac{x}{a^n}\Big)\}$ is ρ-convergent in X_ρ. Hence we may define a mapping $Q : V \to X_\rho$ by

$$Q(x) = \lim_{n\to\infty} a^{4n} f\Big(\frac{x}{a^n}\Big)$$

for all $x \in V$. In fact, this means that

$$\lim_{n\to\infty} \rho\Big(a^{4n} f\Big(\frac{x}{a^n}\Big) - Q(x)\Big) = 0 \tag{3.5}$$

for all $x \in V$. By using the $\triangle_a$-condition with $\triangle_a$-constant k_a, we have

$$\begin{aligned}
&\rho(f(x) - Q(x)) = \rho\Big(f(x) - a^{4n} f\Big(\frac{x}{a^n}\Big) + a^{4n} f\Big(\frac{x}{a^n}\Big) - Q(x)\Big) \\
\leq\ & \rho\Big(\frac{1}{a}\Big(a f(x) - a^{4n+1} f\Big(\frac{x}{a^n}\Big)\Big) + \frac{1}{a}\Big(a^{4n+1} f\Big(\frac{x}{a^n}\Big) - aQ(x)\Big)\Big) \\
\leq\ & \frac{k_a}{a}\, \rho\Big(f(x) - a^{4n} f\Big(\frac{x}{a^n}\Big)\Big) + \frac{k_a}{a}\, \rho\Big(a^{4n} f\Big(\frac{x}{a^n}\Big) - Q(x)\Big) \\
\leq\ & \frac{k_a}{a}\, \frac{1}{k_a^4} \sum_{j=1}^{n} \Big(\frac{k_a^5}{a}\Big)^j \phi\Big(\frac{x}{a^j}, 0\Big) + \frac{k_a}{a}\, \rho\Big(a^{4n} f\Big(\frac{x}{a^n}\Big) - Q(x)\Big)
\end{aligned}$$

for all $x \in V$. As $n \to \infty$, the last part of the above inequalities implies that

$$\rho(f(x) - Q(x)) \leq \frac{1}{a\, k_a^3} \sum_{j=1}^{\infty} \Big(\frac{k_a^5}{a}\Big)^j \phi\Big(\frac{x}{a^j}, 0\Big)$$

for all $x \in V$, that is, the inequality (3.3). Next, we will show the mapping Q is a generalized quartic mapping, that is, it satisfies the equality (1.4) when $b = 1$. We note that

$$\rho\Big(a^{4n} D_a f\Big(\frac{x}{a^n}, \frac{y}{a^n}\Big)\Big) \leq k_a^{4n} \phi\Big(\frac{x}{a^n}, \frac{y}{a^n}\Big) \to 0$$

for all $x,\, y \in V$, as $n \to \infty$.

By an integer number $a \geq 2$ and the Remark 3.1, we have

$$\begin{aligned}
& \rho(D_a Q(x, y)) \\
=\ & \rho\Big(D_a Q(x, y) - a^{4n} D_a f\Big(\frac{x}{a^n}, \frac{y}{a^n}\Big) + a^{4n} D_a f\Big(\frac{x}{a^n}, \frac{y}{a^n}\Big)\Big) \\
\leq\ & \frac{k_a^3}{a^3}\Big[\rho\Big(Q(ax - y) - a^{4n} f\Big(\frac{ax - y}{a^n}\Big)\Big) + \rho\Big(Q(x - ay) - a^{4n} f\Big(\frac{x - ay}{a^n}\Big)\Big) \\
& + \frac{k_a(k_a - 1)^2}{2} \rho\Big(Q(x + y) - a^{4n} f\Big(\frac{x + y}{a^n}\Big)\Big) \\
& - \frac{k_a(k_a + 1)^2}{2} \rho\Big(Q(x - y) - a^{4n} f\Big(\frac{x - y}{a^n}\Big)\Big) \\
& - (k_a^2 - 1)^2 \rho\Big(Q(x) - a^{4n} f\Big(\frac{x}{a^n}\Big)\Big) - (k_a^2 - 1)^2 \rho\Big(Q(y) - a^{4n} f\Big(\frac{y}{a^n}\Big)\Big) \\
& + \rho\Big(a^{4n} D_a f\Big(\frac{x}{a^n}, \frac{y}{a^n}\Big)\Big)\Big]
\end{aligned}$$

for all $x,\, y \in V$. The note and the equation (3.5) imply that $\rho(D_a Q(x, y)) = 0$ for all $x,\, y \in V$. Hence the mapping Q is a generalized quartic mapping, as desired. Finally, we have to show that the mapping Q is unique.

To show the uniqueness, we may assume that there is another a generalized quartic mapping $\widetilde{Q} : V \to X_\rho$ satisfies the inequality (3.3). Then we have $Q(x) = a^{4n} Q(\frac{x}{a^n})$ and $\widetilde{Q}(x) = a^{4n} \widetilde{Q}(\frac{x}{a^n})$ for all $x \in V$. Hence we get

$$\begin{aligned}
\rho(Q(x) - \widetilde{Q}(x)) &= \rho\Big(a^{4n} Q(\frac{x}{a^n}) - a^{4n} \widetilde{Q}(\frac{x}{a^n})\Big) \\
&\leq k_a^{4n} \Big[\rho\Big(Q\Big(\frac{x}{a^n}\Big) - f\Big(\frac{x}{a^n}\Big) - \widetilde{Q}\Big(\frac{x}{a^n}\Big) + f\Big(\frac{x}{a^n}\Big)\Big)\Big] \\
&\leq \frac{2}{a\, k_a^3} \Big(\frac{a}{k_a}\Big)^n \sum_{j=n+1}^{\infty} \Big(\frac{k_a^5}{a}\Big)^j \phi\Big(\frac{x}{a^j}, 0\Big)
\end{aligned}$$

for all $x \in V$. On taking the limit as $n \to \infty$, the uniqueness is proved. $\square$

Corollary 3.6. *Let $a \geq 2$ be an integer number and θ and $p > \log_a \frac{k_a^5}{a}$ be real numbers. Suppose V is a normed space with norm $||\cdot||$ and X_ρ satisfies the $\triangle_a$-condition with $\triangle_a$-constant k_a. If $f : V \to X_\rho$ satisfies*

$$\rho(D_a f(x, y)) \leq \theta(||x||^p + ||y||^p) \tag{3.6}$$

for all $x, y \in V$, then there exists a unique quartic mapping $Q : V \to X_\rho$ such that

$$\rho\Big(f(x) - Q(x)\Big) \leq \frac{\theta\, k_a^2}{a\,(a^{p+1} - k_a^5)} ||x||^p \tag{3.7}$$

for all $x \in V$.

Proof. On taking $\phi(x, y) = \theta(||x||^p + ||y||^p)$ in the Theorem 3.5, we know that the inequality (3.6) holds. Also, the inequalities (3.2) are satisfied. According to Theorem 3.5, the inequality (3.7) holds. $\square$

4. The Fixed Point Method Approach

In this section we use some ideas from [10] and [25] and we shall study the Hyers-Ulam stability for the generalized quartic functional equation (1.4) in a modular space by using the fixed point method. We first assume that ρ is a convex modular on X_ρ with the Fatou property satisfying the $\triangle_a$-condition with $\triangle_a$-constant $0 < k_a < a$, where $a \geq 2$ is an integer.

Theorem 4.1. *Let $\phi : V^2 \to [0,\infty)$ be a function such that there exists an $0 < L < 1$ with*

$$\phi(ax, ay) \le a^4 L\, \phi(x, y) \tag{4.1}$$

for all x, $y \in V$. If $f : V \to X_\rho$ is a mapping satisfying $f(0) = 0$ and

$$\rho(D_a f(x, y)) \le \phi(x, y) \tag{4.2}$$

for all x, $y \in V$, then there exists a unique generalized quartic mapping $Q : V \to X_\rho$ defined by $Q(x) = \lim_{n\to\infty} \frac{1}{a^{4n}} f(a^n x)$ and

$$\rho\Big(f(x) - Q(x)\Big) \le \frac{1}{a^4(1-L)}\phi(x, 0) \tag{4.3}$$

for all $x \in V$.

Proof. We define the set S to be

$$S := \{g : V \longrightarrow X_\rho,\ g(0) = 0\}$$

and define a mapping $\widetilde{\rho}$ on S by

$$\widetilde{\rho}(g) := \inf\{c > 0 \,|\, \rho(g(x)) \le c\,\phi(x, 0)\}$$

for all $x \in V$. By Lemma 3.3 of [25], we have the set S is a linear space, $\widetilde{\rho}$ is a convex modular on S and hence the corresponding modular space $S_{\widetilde{\rho}}$ is the whole space S and is $\widetilde{\rho}$-complete. Moreover, $\widetilde{\rho}$ satisfies the $\triangle_a$-condition with $0 < k_a < a$.

Now, we consider a function $T : S_{\widetilde{\rho}} \to S_{\widetilde{\rho}}$ defined by

$$Tg(x) := \frac{1}{a^4} g(ax) \tag{4.4}$$

for all $g \in S_{\widetilde{\rho}}$ and $x \in V$. Let g, $h \in S_{\widetilde{\rho}}$ be given mappings and $c \in [0, \infty]$ be an arbitrary constant such that $\widetilde{\rho}(g - h) \le c$. The definition of $\widetilde{\rho}$ implies that

$$\rho(g(x) - h(x)) \le c\phi(x, 0)$$

for all $x \in V$. Hence we get

$$\rho\Big(Tg(x) - Th(x)\Big) = \rho\Big(\frac{1}{a^4} g(ax) - \frac{1}{a^4} h(ax)\Big) \le Lc\phi(x, 0)$$

for all $x \in V$. Hence we have

$$\widetilde{\rho}\Big(Tg - Th\Big) \le L\widetilde{\rho}\Big(g - h\Big)$$

for all g, $h \in S_{\widetilde{\rho}}$ such that $\widetilde{\rho}(g - h) \le c$. Hence T is a $\widetilde{\rho}$-stric contraction with a constant L such that $0 < L < 1$. On letting $y = 0$ in (4.2), we have

$$\rho\Big(f(ax) - a^4 f(x)\Big) \le \phi(x, 0) \tag{4.5}$$

for all $x \in V$. On replacing x by ax in (4.5), we have

$$\rho\Big(f(a^2x) - a^4 f(ax)\Big) \le \phi(ax,\, 0)$$

for all $x \in V$. Then

$$\begin{aligned}
&\rho\Big(\Big(\frac{1}{a^4}\Big)^2 f(a^2x) - f(x)\Big) \le \frac{1}{a^4}\rho\Big(\frac{1}{a^4} f(a^2x) - a^4 f(x)\Big)\\
&\le \Big(\frac{1}{a^4}\Big)^2\Big[\rho\Big(f(a^2x) - a^4 f(ax)\Big) + \rho\Big(a^4 f(ax) - (a^4)^2 f(x)\Big)\Big]\\
&\le \Big(\frac{1}{a^4}\Big)^2\Big[\phi(ax,0) + k_a^4\phi(x,0)\Big]\\
&\le \Big(\frac{1}{a^4}\Big)^2\Big[La^4\phi(x,0) + a^4\phi(x,0)\Big]\\
&= \frac{1}{a^4}(L+1)\phi(x,0)
\end{aligned}$$

for all $x \in V$. By the mathematical induction, we have

$$\begin{aligned}
&\rho\Big(\Big(\frac{1}{a^4}\Big)^n f(a^nx) - f(x)\Big)\\
&\le \frac{1}{a^4}\rho\Big(\Big(\frac{1}{a^4}\Big)^{n-1} f(a^nx) - \Big(\frac{1}{a^4}\Big)^{n-2} f(a^{n-1}x) + \Big(\frac{1}{a^4}\Big)^{n-2} f(a^{n-1}x) -\\
&\qquad\qquad \cdots - f(ax) + f(ax) - a^4 f(x)\Big)\\
&\le \frac{1}{a^4}(1 + L + \cdots + L^{n-2} + L^{n-1} + \cdots)\phi(x,0)\\
&= \frac{1}{a^4}\frac{1}{1-L}\phi(x,0)
\end{aligned}$$

for all $n \in \mathbb{N}$ and $x \in V$. Also, we have

$$\begin{aligned}
&\rho\Big(\Big(\frac{1}{a^4}\Big)^n f(a^nx) - \Big(\frac{1}{a^4}\Big)^m f(a^mx)\Big)\\
&\le \frac{1}{a^4}\Big[\rho\Big(a^4\Big(\Big(\frac{1}{a^4}\Big)^n f(a^nx) - f(x)\Big)\Big) + \rho\Big(a^4\Big(\Big(\frac{1}{a^4}\Big)^m f(a^mx) - f(x)\Big)\Big)\Big]\\
&\le \rho\Big(\Big(\frac{1}{a^4}\Big)^n f(a^nx) - f(x)\Big) + \rho\Big(\Big(\frac{1}{a^4}\Big)^m f(a^mx) - f(x)\Big)\\
&\le \frac{2}{a^4}\frac{1}{1-L}\phi(x,0)
\end{aligned}$$

for all $n, m \in \mathbb{N}$ and $x \in V$. This implies that

$$\widetilde{\rho}(T^n f - T^m f) \le \frac{2}{a^4}\frac{1}{1-L}$$

for all $n,\, m \in \mathbb{N}$. Hence we may define

$$\delta_{\widetilde{\rho}}(f) := \sup\{\widetilde{\rho}(T^n(f) - T^m(f)) \,|\, n,\, m \in \mathbb{N}\}\,.$$

By definition of $\delta_{\widetilde{\rho}}(f)$, we may conclude that $\delta_{\widetilde{\rho}}(f) < \infty$. By Lemma 3.3 of [10], the sequence $\{T^n f\}$ is $\widetilde{\rho}$-convergent to $Q \in S_{\widetilde{\rho}}$. Since ρ has the Fatou property, then $\widetilde{\rho}(Tf - f) < \infty$. On letting $x = a^n x$ in (4.5), we have

$$\rho\Big(f(a^{n+1}x) - a^4 f(a^n x)\Big) \leq \phi(a^n x,\, 0)$$

for all $x \in V$. Hence

$$\begin{aligned} \rho\Big(\frac{1}{a^{4(n+1)}} f(a^{n+1}x) - \frac{1}{a^{4n}} f(a^n x)\Big) &\leq \frac{1}{a^{4(n+1)}}\phi(a^n x, 0) \\ &\leq \frac{L^n}{a^4}\phi(x, 0) \leq \phi(x, 0) \end{aligned}$$

for all $x \in V$. Therefore $\widetilde{\rho}(TQ - Q) < \infty$. Hence the limit of $\{T^n f\}$, $Q \in S_{\widetilde{\rho}}$, is a fixed point of the map T; see [10, Theorem 3.4]. Thus we have $\widetilde{\rho}(f - Q) \leq \frac{1}{a^4(1-L)}$, that is, we have

$$\rho(f(x) - Q(x)) \leq \frac{1}{a^4(1-L)}\phi(x, 0)$$

for all $x \in V$. $\square$

Corollary 4.2. *Let $a \geq 2$ be an integer number and ε, θ and $p < 4$ be real numbers. Suppose V is a normed space with norm $||\cdot||$ and X_ρ satisfies the $\triangle_a$-condition with $\triangle_a$-constant k_a. Let L be a constant with $0 < L < 1$. If $f : V \to X_\rho$ satisfies*

$$\rho(D_a f(x,\, y)) \leq \varepsilon + \theta(||x||^p + ||y||^p) \tag{4.6}$$

for all $x,\, y \in V$, then there exists a unique quartic mapping $Q : V \to X_\rho$ such that

$$\rho\Big(f(x) - Q(x)\Big) \leq \frac{\varepsilon}{a^4(1-L)} + \frac{\theta}{a^4(1-L)}||x||^p \tag{4.7}$$

for all $x \in V$.

Proof. On taking $\phi(x,\, y) = \varepsilon + \theta(||x||^p + ||y||^p)$ in the Theorem 4.1, we know that the inequality (4.6) holds. Since L is a constant with $0 < L < 1$ and $p < 4$, we have

$$\begin{aligned} \phi(ax,\, ay) &= a^p\Big(\frac{\varepsilon}{a^p} + \theta(||x||^p + ||y||^p)\Big) \\ &\leq a^4 L\Big(\varepsilon + \theta(||x||^p + ||y||^p)\Big) \\ &= a^4 L\phi(x,\, y) \end{aligned}$$

for all x, $y \in V$. Hence the inequalities (4.1) are satisfied. According to (4.3) of Theorem 4.1, the inequality (4.7) holds. $\square$

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2** (1950), 64-66.

[2] T. Bag and S.K. Samanta, *A comparative study of fuzzy norms on a linear space*, Fuzzy Sets and Sys., **159** (2008), 670-684.

[3] L. Cădariu and V. Radu, *Fixed point methods for the generalized stability of functional equations in a single variable*, Fixed Point Theory and Applications, **2008** (2008), Art. ID 749392.

[4] L. Cădariu and V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Iteration theory (ECIT '02), 43-52, Grazer Math. Ber., 346, Karl-Franzens-Univ. Graz, Graz, 2004.

[5] J.K. Chung and P.K. Sahoo, *On the general solution of a quartic functional equation*, Bulletin of the Korean Mathematical Society, **40** (2003), no. 4, 565-576.

[6] D.H. Hyers, *On the stability of the linear functional equation*, Proceedings of the National Academy of Sciences of the United States of America, **27** (1941), 222-224.

[7] G. Isac and Th.M. Rassias, *Stability of Ψ-additive mappings: Applications to nonlinear analysis*, Internat. J. Math. Math. Sci., **19** (1996), 219-228.

[8] I.H. Jebril and T.K. Samanta, *Fuzzy anti-normed linear space*, J. Math. Tech., (2010), 66-77.

[9] Y.-S. Lee and S.-Y. Chung, *Stability of quartic functional equations in the spaces of generalized functions*, Adv. Diff. Equa., **2009** (2009), Article ID 838347, 16 pages doi:10.1155/2009/838347.

[10] M.A. Khamsi, *Quasicontraction Mapping in modular space without $\triangle_2$-condition*, Fixed Point Theory and Applications, (2008), Artical ID 916187, 6 pages.

[11] H.-K. Kim and H.-Y. Shin, *Refined stability of additive and quadratic functional equations in modular spaces*, J. Inequal. Appl., **2017** (2017), DOI 10.1186/s13660-017-1422-z.

[12] B. Margolis and J.B. Diaz, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc., **126** (1968), 305-309.

[13] D. Miheţ and V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl., **343** (2008), 567-572.

[14] A.K. Mirmostafaee, M. Mirzavaziri and M. S. Moslehian, *Fuzzy stability of the Jensen functional equation*, Fuzzy Sets and Systems, **159** (2008), 730-738.

[15] A.K. Mirmostafaee and M.S. Moslehian, *Fuzzy versions of Hyers-Ulam-Rassias theorem*, Fuzzy Sets and Systems, **159** (2008), 720-729.

[16] H. Nakano, *Modulared Semi-Ordered Linear Spaces*, Maruzen, Tokyo, 1950.

[17] C. Park, *Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras*, Fixed Point Theory Appl., **2007** (2007), Art. ID 50175.

[18] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory, **4** (2003), 91-96.

[19] J.M. Rassias, *Solution of the Ulam stability problem for quartic mappings*, Glasnik Matematicki Series III, **34** (1999), 243-252.

[20] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297-300.

[21] I.A. Rus, *Principles and Applications of Fixed Point Theory*, Ed. Dacia, Cluj-Napoca, 1979 (in Romanian).

[22] P.K. Sahoo, *A generalized cubic functional equation*, Acta Math. Sinica, **21** (2005), 1159-1166.

[23] S.M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, 1960.

[24] T.Z. Xu, J.M. Rassias, and W.X. Xu, *A generalized mixed quadratic-quartic functional equation*, Bull. Malaysian Math. Scien. Soc., **35** (2012), 633-649.

[25] K. Wongkum, P. Kumam, Y.J. Cho, and P. Chaipumya, *On the generalized Ulam-Hyers-Rassias stability for quartic functional equation in modular spaces*, J. Nonlinear Sci. Appl., **10** (2017), 1-10.

*

College of Liberal Arts,
Dankook University, 152, Jukjeon, Suji,
Yongin, Gyeonggi, 16890, Republic of Korea
E-mail: `khjmath@dankook.ac.kr`